Towards a method for solving partial differential equations by using wavelet packet bases

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Abstract

In this paper we present a new methodology based on the wavelet packet concept, in order to define an adaptive method for the approximation of partial differential equations. The wavelet packet framework allows us to define the notion of a minimal basis that has proven to be an efficient procedure for data compression. The purpose here is to take benefit of this compression to represent accurately and economically the solution of a time dependent PDE. The time discretization is a standard multistep scheme. The spacial discretization is defined by inferring a reduced basis for the solution at the new time step, from the knowledge of the previous ones.

1. Introduction

The theory of orthonormal wavelet bases has been developed by Meyer [1] and his group. It allows an efficient representation to be given to characterize isolated defects of some (otherwise regular) function, thanks to a particularly good localization both in space and scale of each element of the basis. The decomposition of such a function is lacunary, in the sense that very few coefficients of its decomposition on a wavelet basis are non-negligible. This allows compressed approximations to be defined by getting rid of the coefficients that are smaller than a prescribed threshold. The natural applications of these bases are in image analysis and data compression. The next candidate for application of this theory is to define efficient adaptive methods for the approximation of non-stationary partial differential equations. We refer for example, to [2, 3] for more details on the definition of the algorithms and their implementation, and to [4] for some details on the numerical analysis of the method. The central idea in the previous papers is to infer the position or the creation of the (isolated) defect of the solution at some time step by knowing the wavelet decomposition of the solution at the previous time steps. This adaptive method is promising in the case where one can predict that the defect in the solution (sharp gradients or discontinuities) are the exception. Indeed, with application of this method to approximate the solution of the one-dimensional Burger's equation, a reduction factor of one-eighth has been observed for the number of degrees of freedom, with respect to a standard uniform representation. This reduction factor is maintained, in each dimension, in multidimensional examples.

The domain of application of this type of adaptive method is certainly reduced and cannot be used in approximations where defects are not the exception. This is the case for instance if one wants to simulate turbulent flows. In this direction, one can notice the recent work of Farge et al. [5] that compares the analysis of a turbulent field by spectral and wavelet packet decompositions. The previous reduction factor in representative coefficients is observed for the wavelet packet decomposition. The concept of wavelet packets has been introduced by Coifman et al. [6] as a generalization of the wavelet

bases. It relies on the definition of a library of orthonormal bases depending on three parameters (space, scale and frequency) of the same dimensional space, and a procedure to choose, among all of these, the best candidate for the most economical way to represent a given function. This best basis is chosen by minimizing some given entropy attached to the coefficients in each basis of the library.

In this paper, we generalize the notion of an adaptive method to the wavelet packet framework, in the same spirit as [2, 3]. In Section 2, we recall the main notion on wavelet packets required for the paper and define the basics of the adaptive algorithm introducing the concept of neighbors. In Section 3, we detail this notion of neighbouring elements in space, scale and frequency. Finally, in Section 4, we illustrate the potentiality of the method by several numerical examples.

2. Presentation of the method

For the sake of simplicity, we present the basics of the method applied to the case where periodic boundary conditions are prescribed to the solution of a partial differential equation. This allows us to skip the treatment of some boundary conditions of Dirichlet or Neumann type. Let T be the torus $[0, 1]$. The wavelet framework involves at the very beginning the decomposition of the Lebesgue space $L^2(T)$ into the scale of increasing finite dimensional subsets:

$$V_0 \subset V_1 \subset \cdots \subset V_p \subset \cdots \subset L^2(T) .$$

Each finite dimensional space V_p has dimension 2^p and is provided with a library of orthonormal bases, that are constructed from the elements $w_{n,k}^j$ defined as

$$\forall x \in \mathbb{R} \quad w_{n,k}^j(x) = 2^{j/2} w_n(2^j x - k) , \quad 0 \leqslant j \leqslant p , \ 0 \leqslant n \leqslant 2^{p-j} - 1 , \ 0 \leqslant k \leqslant 2^j - 1 .$$

The elements w_n are given in [6] and ordered by increasing frequency as explained in [7]. In the previous notation, j is a scaling index that changes by dilation of the elements of the basis, n is the frequency index and finally, k is the position index that changes by translation. In particular, for any fixed j, the set

$$\mathcal{B}_j = \{w_{n,k}^j, 0 \leqslant n \leqslant 2^{p-j} - 1, 0 \leqslant k \leqslant 2^j - 1\} ,$$

is a basis of V_p, but many other bases can be constructed by associating these elements in different ways [6].

Among all the possible choices, the 'best one' for some particular function φ_p of V_p, will be the one in which some entropy is minimized. Here, the entropy we use is the number of coefficients in the expansion of φ_p larger than a prescribed threshold ϵ. These coefficients will correspond to the degrees of freedom for the approximation of the solution; this legitimates our choice of entropy (see Section 4). Note that our definition does not coincide with the standard one, which allows us to optimize some other criteria of the representation [7].

The wavelet packets that are used in this paper are those associated with the cubic spline functions of T. The torus $[0, 1]$ is decomposed into 2^p equidistant points, the space V_p is then the set of cubic spline functions associated with this partition.

REMARK 2.1. The definition of the best basis for a function is not a continuous procedure. Indeed, let ψ be close to φ in $L^2(T)$. Then the respective best bases may be quite different. Nevertheless, the entropy of the decomposition of ψ in the best basis associated with φ (denoted as $\mathcal{B}(\varphi)$) is only slightly larger than the entropy of the decomposition of ψ in its best basis $\mathcal{B}(\psi)$. Hence, it is reasonable to use $\mathcal{B}(\varphi)$ to decompose ψ. It can also be noticed that the basis functions in $\mathcal{B}(\varphi)$ that are really representative of ψ (i.e. the corresponding coefficient is $> \epsilon$) can be recognized as 'neighbours' which are really representative of φ; the notion of neighbours will be clarified in Section 4.

NOTATION 2.2. The approximation of a given function $\varphi \in L^2(T)$ by an element of V_p relies on three steps:

(i) projection of φ in V_p resulting in φ_p;
(ii) definition of the best basis associated with φ_p, i.e. $\mathcal{B}(\varphi_p)$ and the corresponding set of indices $A(\varphi_p) = \{(j,n,k), \omega^j_{n,k} \in \mathcal{B}(\varphi_p)\}$ so that

$$\varphi_p = \sum_{(j,n,k)\in A(\varphi_p)} \alpha^j_{n,k} \omega^j_{n,k} ;$$

(iii) filtering procedure that consists of getting rid of the coefficients smaller than ϵ, i.e. from $A(\varphi_p)$, we define $\tilde{A}(\varphi_p)$ as

$$\tilde{A}(\varphi_p) = \{(j,n,k) \in A(\varphi_p), |\alpha^j_{n,k}| > \epsilon\} .$$

The approximation of φ is then given by $\tilde{\varphi}_p = \Sigma_{(j,n,k)\in\tilde{A}(\varphi_p)} \alpha^j_{n,k} \omega^j_{n,k}$. With these notations, we easily obtain the following lemma.

LEMMA 2.3. For any function $\varphi \in L^2(T)$,

$$\|\varphi - \tilde{\varphi}_p\|_{L^2(T)} \leq \inf_{\psi_p \in V_p} \|\varphi - \psi_p\|_{L^2(T)} + \sqrt{2^p}\epsilon$$

The two previous statements lead naturally to the following numerical algorithm for the approximation of the solution of some partial differential equation. As is standard, the model equation is the Burgers' equation with periodic boundary conditions: Find u, such that

$$\frac{\partial u}{\partial t} + \frac{1}{2} u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} , \quad t \geq 0 , \ x \in [0,1] , \tag{1}$$

with initial condition

$$u(0,x) = u^0(x)$$

and viscosity parameter equal to $\nu = 1/400\pi$. The approximation involves a discrete parameter in time Δt that is a positive real number, and a discrete parameter in space p. The solution $u(m\,\Delta t, \cdot)$ is approximated by an element $u^m = u^m_p$ of V_p defined by the following algorithm.

ALGORITHM

Initial step. Let $A(u^0_p)$ be the indexes of the best associated to u^0_p, $\tilde{u}^0_p$ and $\tilde{A}(u^0_p)$ stand as in Notation 2.2. Set $A^0 = \tilde{A}(u^0_p)$.

$(m+1)$*th step*. Let u^m be given by

$$u^m = \sum_{(j,n,k)\in A^m} \alpha^{j,(m)}_{n,k} \omega^j_{n,k} ,$$

where $A^m \subset A(u^0_p)$.

(i) Define first $\tilde{u}^m$ and $\tilde{A}^m$ as explained in Notation 2.2(iii).
(ii) Define A^{m+1} as follows:

$$A^{m+1} = \{(j,n,k) \in A(u^0_p) \text{ that are neighbours of } (j',n',k') \in \tilde{A}^m\} ,$$

where the definition of neighbours has been suggested in Remark 2.1 and will be detailed in Section 3.

(iii) Compute u^{m+1} written as

$$u^{m+1} = \sum_{(j,n,k)\in A^{m+1}} \alpha^{j,(m+1)}_{n,k} \omega^j_{n,k}$$

and such that it satisfies the discrete problem

$$\forall (j,n,k) \in A^{m+1} \quad \int_0^1 \frac{u^{m+1} - \tilde{u}^m}{\Delta t} w^j_{n,k}\, dx + \nu \int_0^1 \frac{du^{m+1}}{dx} \frac{dw^j_{n,k}}{dx} dx = -\frac{1}{2} \int_0^1 \frac{d(\tilde{u}^m)^2}{dx} w^j_{n,k}\, dx .$$

The time discretization that is presented here is the simplest one and, of course can be easily improved, but this is not the point of this paper.

REMARK 2.4. The previous algorithm will certainly lead to increase in the dimension of A^{m+1} and in order to minimize the basis for the representation of u, we certainly have to re-compute the best basis of u^m from time to time and use the set of indexes $A(u^m)$ in place of $A(u_p^0)$ for further time steps.

3. Topology of the wavelet packets in the position-frequency space

As explained in the previous section, the adaptive procedure is based on the notion of neighbours. This has to be well suited in order to be able to fit, at best, the evolution of the elements of the best basis that are important to represent the solution well. Let us indicate now that each element $w_{n,k}^j$ is associated with a particular point of discretization $x_l = l/2^p$, $0 \leq l \leq 2^p - 1$. The choice of best basis induces numbering of these points in a particular way, since each element $w_{n,k}^j$ is associated with a 'centre' $x_l = x_{l(j,n,k)}$ as will be detailed in [8].

Let $(j_0, n_0, k_0) \in A(u_p^0)$ be a particular index, the neighbours of which have to be determined. Recall that the frame within which the neighbours will be chosen, is the best basis $B(u_p^0)$ of u_p^0. In a preliminary stage, the elements $w_{n,k}^j$, $(j, n, k) \in A(u_p^0)$ are grouped by Fourier localization [7] in sets b_q, $0 \leq q \leq 2^p - 1$,

$$b_q = \{w_{n,k}^j, (j, n, k) \in A(u_p^0), n2^j = q\} .$$

Note that some of the b_q can be empty, the others group elements of $A(u_p^0)$ localized around the same point in Fourier space.

These two considerations allow for defining, in a position-frequency diagram, the influence rectangle of each element $w_{n,k}^j$ of the basis, centred around the point $(x_{l(j,n,k)}, n2^j)$; see for instance the representation in Fig. 1.

Let $q_0 = n_0 2^{j_0}$ be the Fourier localization of $w_{n_0,k_0}^{j_0}$. The neighbours of this basis function are those elements of $A(u_p^0)$,

– either in the same group b_{q_0}: $w_{n_0,k_0-1}^{j_0}$ and $w_{n_0,k_0+1}^{j_0}$ the centre of which is on both sides of $x_{l(j_0,n_0,k_0)}$,

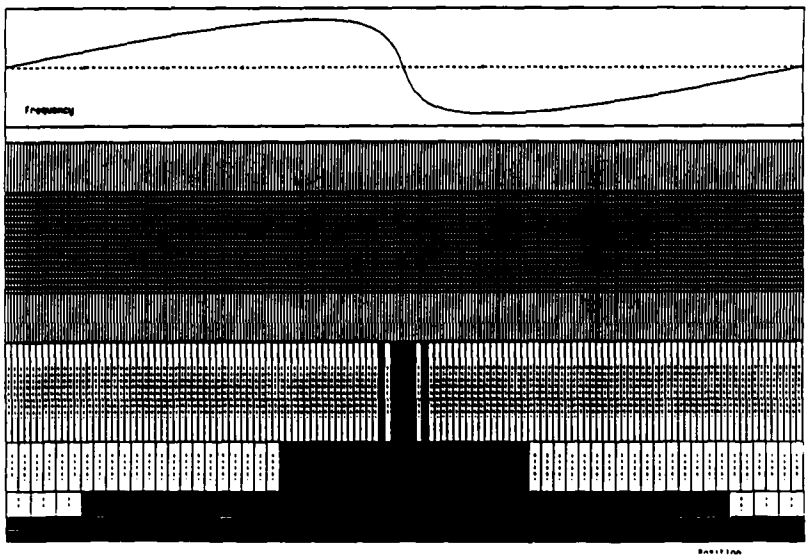

Fig. 1. Representation in a position-frequency diagram of the best basis associated with the function plotted at the top. The black rectangles are those for which the related coefficients are $>10^{-6}$.

- or in the previous (non-empty) group $b_{q'}$: corresponding to the one or both elements $w^j_{n,k}$ the centre of which is (are) the closest to $x_{l(j_0,n_0,k_0)}$,
- or in the next (non-empty) group $b_{p''}$: corresponding to the one or both elements $w^j_{n,k}$ the centre of which is (are) the closest to $x_{l(j_0,n_0,k_0)}$.

We can notice that the number of neighbours is $\leqslant 6$. Nevertheless, this induces a small increase in dimension between $\tilde{A}^m$ and A^{m+1} since most of the neighbours are already elements of $\tilde{A}^m$.

In order to check that the notion of neighbours we have defined is accurate, we perform some preliminary numerical experiments in the last section. The global algorithm will be reported in [8].

4. Numerical experiments

4.1. Comparison of the entropy condition

Recall first that our choice of entropy condition is in order to minimize the number of degrees of freedom to represent the solution. The definition we have introduced involves a threshold parameter $\epsilon = 10^{-6}$ and differs from the standard entropy of [7]. In order to justify our definition, we compare the two entropies and also the plain wavelet approximation. The comparison is performed on the evolution of the solution of the viscous Burgers' equation (computed by some accurate scheme). The analysis is done in the space V_p with $2^p = 512$.

The first results are related to the solution with initial condition equal to $u^0(x) = \sin(2\pi x)$ that develops only one sharp gradient at time $t = 2/\pi$. In Fig. 2(a) we have plotted the evolution in time of the number of coefficients of the solution larger than the threshold ϵ for the three bases. It is immediate to notice that our definition of the entropy induces a net win in the number of degrees of freedom. Note also that the plain wavelet representation is as good as the wavelet packet one. This is in agreement with our discussion in the introduction.

The next example confirms the previous conclusion as regards the choice of entropy and justifies the potential win of the wavelet paquet with respect to the plain wavelet representation in the case where the initial condition induces many singular regions. In this case, the initial condition is $u^0(x) = \sin(8\pi x)$ with the same amount of viscosity that leads to four sharp gradients.

Note also that in all cases, the error in the L^2-norm behaves similarly and is about 3ϵ.

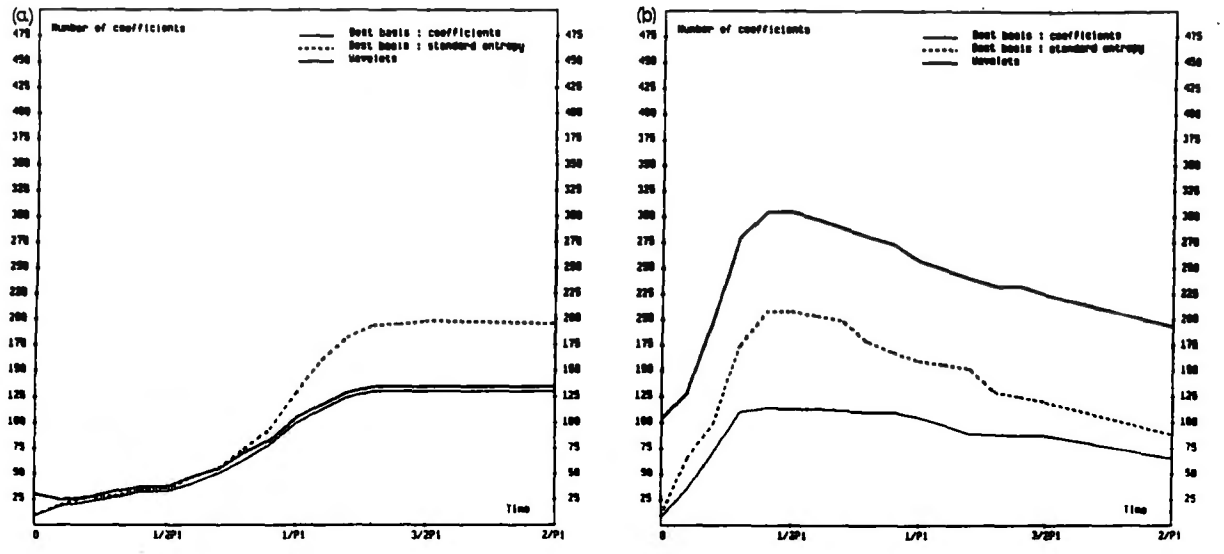

Fig. 2. Evolution of the number of coefficients larger than 10^{-6} for the solution of Burger's equation: (a) with initial condition $u^0(x) = \sin(2\pi x)$, (b) with initial condition $u^0(x) = \sin(8\pi x)$.

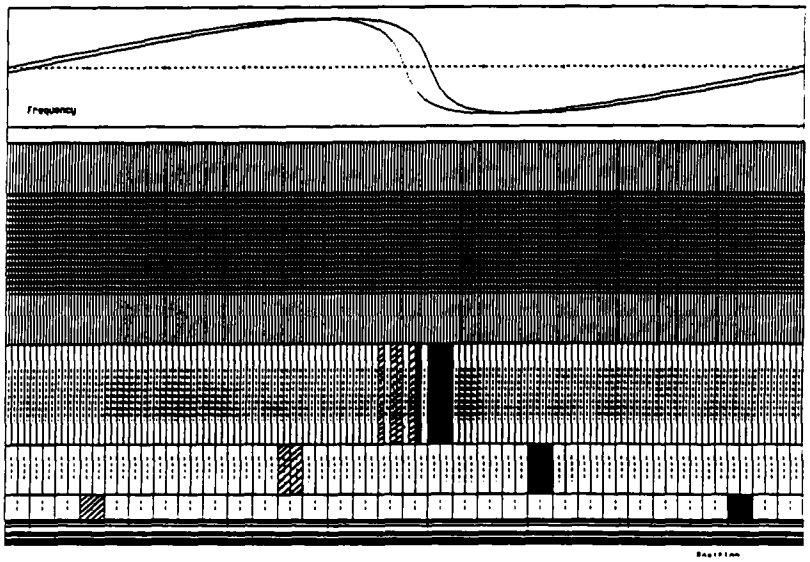

Fig. 3. Stability of the best under translation: the rectangles symbolize the position-frequency location of basis functions which represent the solution; the black ones are those required by the translated solution and not by $u_{0,9}$ while the grey ones are those used by $u_{0,9}$ and not by the translated solution.

REMARK 4.1. It is interesting to notice also that in both figures, the number of degrees of freedom increases initially when the effects of non-linear terms are predominant and then (between $t = 1/\pi$ and $t = 2/3\pi$ in Fig. 2(a) and slightly before $t = 1/2\pi$ in Fig. 2(b)) the viscosity effects retain the sharpening of the gradients.

4.2. Stability of the best basis

The stability of the best basis is illustrated here for both translation and sharpening of gradients. We have first defined the best basis $B_{0,9}$ of the solution $u_{0,9}$ at time $t = 9\pi/10$ for the Burgers' equation with initial condition $u^0(x) = \sin(2\pi x)$. Then we decompose $u_{0,9}(x - \pi/10)$ in B_9. The number of coefficients larger than ϵ is about the same as for the original solution although the best basis is different; in

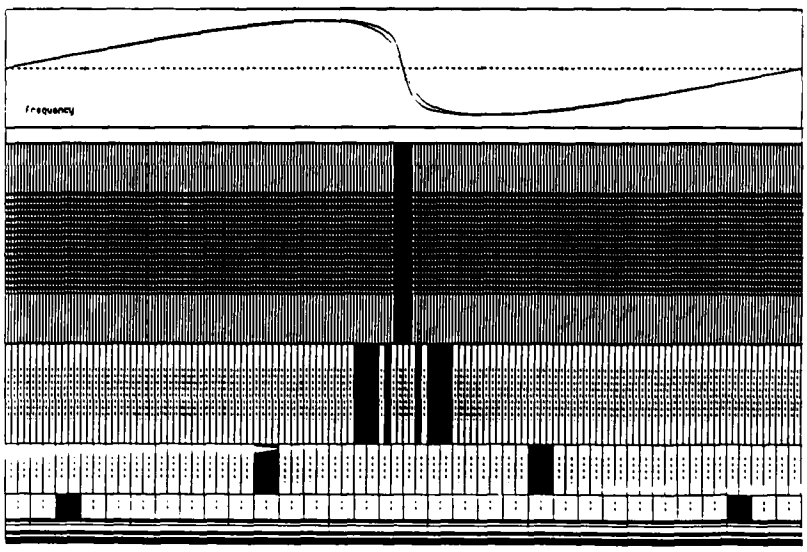

Fig. 4. Stability of the best basis under sharpening of gradients.

addition, the coefficients involved are all neighbours in the position of those of $u_{0.9}$. The same conclusion holds for the analysis of the solution u_1 of the Burgers equation at time $t = \pi$ that presents a larger gradient than $u_{0.9}$: the coefficients which appear in the decomposition are neighbours in space and frequency to those of $u_{0.9}$. Note that the difference in time between the two functions $u_{0.9}$ and u_1 corresponds to 100 Δt in our experiment. These results are plotted in Figs. 3 and 4, respectively. This confirms the stability of the best basis for these two types of evolution.

These two experiments show the viability of our approach, based on the concept of neighbours we have introduced. More experiments are in progress to prove that such a Galerkin approach leads to an effective numerical scheme.

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